

Distinguishing J=4 from J=0 on a cubic lattice

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1. THE PROBLEM

On a square lattice (we consider here a D=2+1 lattice for simplicity) the exact rotational symmetry is confined to rotations that are integer multiples of $\pi/2$. If we attempt to build operators of spin J from basic components that are identical – that is, related by a lattice symmetry – then there is an ambiguity:

$$\exp(iJ\theta) \equiv \exp(iJ'\theta) \quad \forall \theta = \frac{n\pi}{2} \quad (1)$$

if

$$J' = J + 4N \quad \forall N \quad (2)$$

Thus, for example, we cannot distinguish J=0 from J=4.

It is conventional to assign the lowest value of continuum J to such an operator. However even if this choice is correct for the lightest state of a given ‘J’ – see the discussion in [1] – the ambiguity cannot be avoided once we start calculating several excitations of the same ‘J’.

Of course this is a self-inflicted problem. As $a \rightarrow 0$, we know that we recover full rotational invariance on physical length scales. Therefore we should be able to construct operators that become as close as we like to spin J, for any J, as $a \rightarrow 0$. This will require using linear combinations of operators that are *approximate* rotations of each other, where the approximation is such that it becomes exact as $a \rightarrow 0$.

In this poster we are going to explore how well the simplest embodiment of this idea works. We shall do so by attempting to calculate the J=0 and J=4 glueball masses in the 2+1 dimensional SU(2) lattice gauge theory.

2. FIRST STEPS ...

We start with some closed loop on the lattice that is symmetric about the x-axis. Call it ϕ_{Ax} . There is a corresponding loop about the y-axis, ϕ_{Ay} , which is identical in the sense that e.g.

$$\langle \phi_{Ay}(t) \phi_{Ay}(0) \rangle = \langle \phi_{Ax}(t) \phi_{Ax}(0) \rangle \quad \forall t \quad (3)$$

If we sum these two, $\phi_A = \phi_{Ax} + \phi_{Ay}$, then we obtain an operator ϕ_A that we would normally call ‘J=0’. (As usual we always take zero-momentum sums of such operators.)

We now construct a loop that is symmetric around the diagonal which is at $\pi/4$ to the x-axis. We choose a loop that ‘looks’ as though it is roughly a rotation of ϕ_{Ax} . There will be an identical loop that is rotated by $\pi/2$. Summing these two loops gives us the diagonal operator ϕ_D .

Clearly our trial J=0 operator is going to be

$$\phi_{J=0} = \phi_A + c\phi_D \quad (4)$$

and our trial J=4 operator is going to be

$$\phi_{J=4} = \phi_A - c\phi_D \quad (5)$$

(since $\exp(iJ\theta) = -1$ for J=4 and $\theta = \pi/4$.)

If ϕ_A and ϕ_D were indeed identical (up to a rotation) then we would take $c=1$ and then our ‘J=0’ state would contain no J=4 and vice-versa. However they are not identical and so we will choose c so as to enhance whatever approximate identity they possess e.g.

$$c^2 \langle \phi_D(t) \phi_D(0) \rangle \simeq \langle \phi_A(t) \phi_A(0) \rangle \quad \forall t \quad (6)$$

We now give the simplest example of such operators. Let U_j be the matrix on the link in direction j . I suppress the site index. Then our

example consists of the following path ordered products

$$\phi_A = Tr\{U_x U_x U_y U_x^\dagger U_x^\dagger U_y^\dagger\} \quad (7)$$

and

$$\phi_D = Tr\{U_x U_y U_x U_y U_x^\dagger U_y^\dagger U_x^\dagger U_y^\dagger\} \quad (8)$$

with, in each case, the addition of the operator that is rotated by an angle of $\pi/2$. So the basic on-axis operator is a 1×2 rectangular loop, and the corresponding ‘diagonal’ operator is an ordered product of 2 plaquettes. We can obviously extend this to $1 \times n$ rectangular loops and to diagonal operators that are ordered products of n plaquettes.

This operator is only useful if ϕ_D is close to being identical to ϕ_A (up to a normalisation) in which case the operators are close to being $\pi/4$ rotations of each other and we can form good $J=0$ and $J=4$ combinations. In Fig.1 we compare these operators, using ‘twice-blocked’ link matrices [2]. (The normalisation c in eqns(4,5) has been chosen so that the vacuum-subtracted correlation functions at $t = 0$ are equal.) We observe that the on-axis and diagonal operators are indeed ‘close’ to being identical and so provide at least some kind of starting point for distinguishing $J = 0$ from $J = 4$.

3. A CALCULATION

As a starting basis of operators we use 1×2 , 1×3 , 1×4 and 2×4 loops, with the diagonal loops chosen in the obvious way. We construct these loops not just out of the elementary lattice links but also out of blocked links (to enhance the overlaps onto the lightest states). In practice we shall use once, twice and thrice blocked links so that our initial basis contains $4 \times 3 = 12$ operators. As seen above, choosing the constant c in eqns(4,5) so that

$$c^2 \langle \Phi_D(t) \Phi_D(0) \rangle = \langle \Phi_A(t) \Phi_A(0) \rangle \quad (9)$$

for the vacuum-subtracted operators, with $t=0$ (or indeed $t=a$), works quite well. We will, from now on, choose to use $t=0$, and all the operators we use shall be vacuum subtracted and normalised so the $t = 0$ correlators are unity.

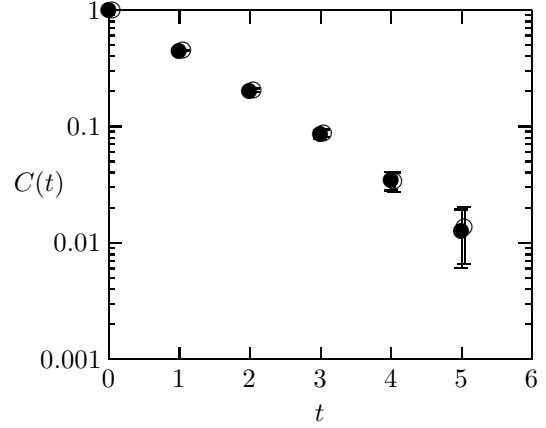


Figure 1. On-axis (●) and diagonal (○) 1×2 loop vacuum-subtracted correlations.

The exploratory calculations shown here are obtained on a modest run of 5000 sweeps on a 24^3 lattice at $\beta = 9$ in the D=2+1 SU(2) lattice gauge theory.

If our operators really were $J=0$ and $J=4$ – which can only occur in the continuum limit – then they would be mutually orthogonal:

$$\langle \Phi_{J=0}(0) \Phi_{J=4}(0) \rangle = 0. \quad (10)$$

We calculate these overlaps and find that the worst operator (in the sense of having the largest such overlap) is the one based on the 1×2 loop. Hence we discard it from now on, so that our basis is reduced to 9 operators. For these operators we find that the overlaps in eqn(10) are very small compared to the overlaps $\langle \Phi_{J=0} \Phi'_{J=0} \rangle$. This confirms that our procedure is working reasonably well.

How do we extract a mass? We know from positivity that

$$\langle \Phi_J(t) \Phi_J(0) \rangle \leq e^{-m_J t} \quad (11)$$

where m_J is the lightest mass (or energy).

So given a set of operators Φ_J^i ; $i=1, \dots, N$, a variational estimate of the mass is provided by

$$m_J = \min_i m_{J,eff}^i(t) \quad (12)$$

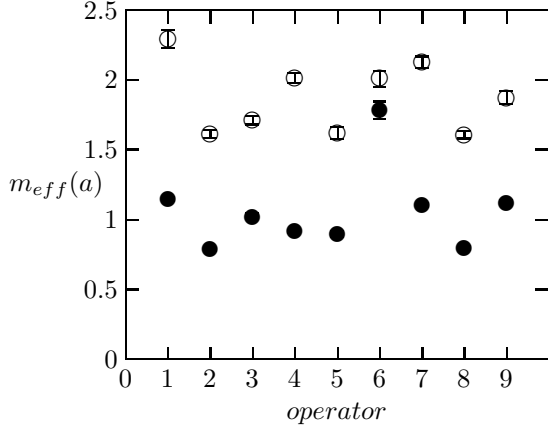


Figure 2. $t = a$ effective masses from $J = 0$ (\bullet) and $J = 4$ (\circ) correlators.

where we have defined a set of effective masses by

$$m_{J,eff}^i(t) = -\frac{1}{t} \ln \{ \langle \Phi_J^i(t) \Phi_J^i(0) \rangle \}. \quad (13)$$

In Fig.2 we show the values of $m_{eff}^i(t = a)$ for our nine $J=0$ and $J=4$ operators. We use $t=a$ both because the effective masses become rapidly less accurate if we increase t ; and also because we know that at large enough t the lightest scalar mass will dominate. That is to say, a good ‘ $J=4$ ’ operator will have two quite distinct effective mass plateaux: the one at lower t will give $m_{J=4}$ while the one at larger t will give $m_{J=0}$ (if it is lighter). As $a \rightarrow 0$ the first plateau will extend to ever larger values of t . In this preliminary test of the method we do not attempt to illustrate such behaviour.

In our plot we see a distinct separation between the $J=0$ and $J=4$ effective masses at $t=a$ and, applying the above variational criterion, we obtain the following mass estimates (in lattice units):

$$m(J = 0) = 0.786(12) \quad (14)$$

$$m(J = 4) = 1.607(27) \quad (15)$$

The $J = 0$ mass is close to the expected value [1]. Is the lightest $J = 4$ state a glueball or a state composed of 2 $J = 0$ glueballs in an $L = 4$ partial

wave? The latter is unlikely: the $L = 4$ suppression near the branch point of the cut should lead to an effective mass that is considerably above the energy at the branch point $E = 2m(J = 0)$. In addition we expect our single trace operators to have a small projection on states composed of two colour singlets. Evidence for these expectations comes from earlier numerical work [3]. Although all this will eventually need explicit investigation, in this preliminary study we assume it to be so and that what we have calculated is the lightest $J = 4$ glueball mass.

A better variational calculation would be to work with all linear combinations of the $N(=9)$ Φ_J^i . This is discussed in [4].

Of course, simple as all this is in principle, in practice it rapidly becomes something of a Heath Robinson construction if one wants to extend it to arbitrary J . What one wants is a systematic, general procedure that does not require tailoring for the individual ‘bulges’ of each J . For example one can proceed as follows. As $a \rightarrow 0$ the density of lattice sites within two concentric circles of radii r and r' , where these radii are fixed in physical units, becomes arbitrarily dense. This will even be the case if we choose $r \rightarrow r'$ in physical units, as long as $|r - r'|/a \rightarrow \infty$. In that case we can construct triangular operators (constructed out of some well-chosen gluonic propagators connecting the centre to 2 sites between r and r') that become rotations of each other for arbitrary angles of rotation as $a \rightarrow 0$. These can then be used as part of a systematic procedure for constructing operators of arbitrary spin J as $a \rightarrow 0$. This approach is described in [4].

REFERENCES

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